Mice and Poisoned Wine Problem

Stephane Moore

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1] Problem

You have just commissioned a shipment of n immense bottles of wine. Upon receipt you are informed that one of the bottles of wine has been poisoned. You are given limited information regarding the poison: ingestion in any quantity is fatal and no other detection methods exist. Your supervisor has permitted you a request for a minimal number of mice in order to isolate the lethal vintage. You have also been reminded that dilly-dallying will not be excused. As such, anything you want tasted will have to be prepared in advance. How many mice should you request?

2] General Solution

Let us define the literals w and m to respectively denote a finite enumeration of $n \in \mathbb{N}^*$ wine bottles and a finite enumeration of $k \in \mathbb{N}^*$ mice at your disposal.
We will use the following definition to describe the state of w:

$$p(i) = \begin{cases} 1 & \text{if } i\text{th bottle } w_i \text{ is poisoned;} \\ 0 & \text{otherwise.} \end{cases}$$

The most important property of this matrix is that $\sum_{i=1}^{n} p(i) = 1$.
The naive approach here would be something akin to assigning a mouse to each bottle of wine. Although this would certainly minimize casualties, that is not your prerogative. You are to minimize the number of rodents involved in this operation. Let us consider another approach to the problem.
Firstly, we should note that since there is only one poisoned bottle, there are only n possible states of the set w. Thus the state of w can be represented by the domain $[1, n] \cap \mathbb{N}^*$.
Let us define m to be an enumeration of k mice. The vitality of a mouse m_i is identified as follows:

$$v(i) = \begin{cases} 1 & \text{if } i\text{th mouse } m_i \text{ is dead;} \\ 0 & \text{otherwise.} \end{cases}$$

Essentially, the enumeration m represents a domain $\{0, 1\}^k$.
Consider an injective function $f : ([1, n] \cap \mathbb{N}^*) \to \{0, 1\}^k$ and a surjective function $g : \{0, 1\}^k \to ([1, n] \cap \mathbb{N}^*)$ that satisfy the property $g(f(x)) = x$. Note that f is not necessarily an invertible function.
For each wine w_i, its assignment amongst the k mice can be generated by the function $f(i)$. For the poisoned wine bottle w_p where $p \in ([1, n] \cap \mathbb{N}^*)$, we know that $g(f(p)) = p$. $f(p)$ identifies which mice drank w_p. Any mouse that drinks w_p dies, thus $f(p)$ also describes which mice die. The mice that die are identified after we perform the wine tasting; thus $f(p)$ is a given. Using the function g we are easily able to determine w_p from $g(f(p)) = p$.
Given the injective function f, we know that $f(x) = f(y) \implies x = y$. The domain of f is $[1, n] \cap \mathbb{N}^*$; thus there are n values in the domain. The injective property requires that there be at least n values in the range of f. If the range of f is $\{0, 1\}^k$, then the following is true:

$$\begin{aligned} |\{0, 1\}|^k &\geq n \\ 2^k &\geq n \\ log_2(2^k) &\geq log_2(n) \\ k &\geq log_2(n) \end{aligned}$$

Hence the minimal value of $k \in \mathbb{N}^*$ is $k = \lceil log_2(n) \rceil$. Thus it follows that the minimum number of mice needed to identify the poisoned wine bottle is $k = \lceil log_2(n) \rceil$.

3] Example Solution

Let us now define an injective function $f_b : ([1, n] \cap \mathbb{N}^*) \rightarrow \{0, 1\}^k$ and a surjective function $g_d : \{0, 1\}^k \rightarrow ([1, n] \cap \mathbb{N}^*)$ that satisfy the property $g_d(f_b(x)) = x$.
Let us consider that $f_b(x)$ evaluates to some $(m(x, k), m(x, k-1), ..., m(x, 1))$ where each $m(x, i) \in \{0, 1\}$. Let us construct $f_b(x)$ as follows:

$$\begin{aligned} m(x, i) &= \frac{x}{2^{i-1}} \ mod \ 2 \\ f_b(x) &= (m(x, k), m(x, k-1), ..., m(x, 1)) \end{aligned} \tag{1}$$

We see that f_b simply converts x to its digits in binary. According, let us construct the function g_d as:

$$g_d(\ (m(x, k), m(x, k-1), ..., m(x, 1) \ )) = \sum_{i=1}^{k} \left[m(x, i) * 2^{i-1} \right]$$

Here, g_d simply calculates the value represented by the binary digits $m(x, k)m(x, k-1)...m(x, 1)$. Thus f_b converts $x \in \mathbb{N}^*$ to binary digits and g_d converts the binary digits back to an integer. Intuitively we see that $g_d(f_b(x)) = x$. For this pair (f_b, g_d) the minimal value of k is the number of bits need to represent the n values. From before we know that this is $k = \lceil log_2(n) \rceil$. Thus f_b and g_d are functions that allow us to identify the poisoned wine bottle with $k = \lceil log_2(n) \rceil$ mice.